Part 1
Probabilistic Patterns of Univariate Statistical Extremes

Annex 1
On The “Duality” between Extremes and Sums

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For simplicity we will deal with some “duality” between sums (or averages) and maxima, the translation to minima being obvious from the relation
\[
\min \{X_i\} = -\max \{-X_i\}.
\]

The “duality” is expressed by the two columns in correspondence, where there are various gaps. \(F(\cdot), F(\cdot\cdot\cdot)\) and \(\varphi(\cdot), \varphi(\cdot\cdot\cdot)\) will denote the distribution functions and the characteristic functions.
\[
S_k = \sum_{i=1}^{k} X_i
\]
\[
\varphi_X(t) = M_X(e^{itX}): \text{ch. f. of } X
\]
\[
X_i \text{ indep.: } \varphi_{S_k}(t) = \prod_{i=1}^{k} \varphi_i(t)
\]
\[
X_i \text{ i.i.d.: } \varphi_{S_k}(t) = \varphi^k(t)
\]
\[
(X, Y) \text{ indep: } \varphi_{aX + bY}(t) = \varphi_X(a \ t) \varphi_Y(b \ t)
\]
\[
M(aX + b) = a M(X) + b
\]
\[
V(aX + b) = a^2 V(X)
\]
If \{X_i\} i.i.d. have \(\mu, \sigma^2\) then
\[
\varphi(t) \xrightarrow{(s_k - k\mu)/\sqrt{k} \sigma} (e^{-t^2/2})
\]
(Central Limit Theorem); in the general case “sometimes” the ch. f. of the normal law may be substituted by that of an indefinitely divisible law; \(\mu = \varphi_X(0)/i, \sigma^2 = \varphi_X(0)^2 - \varphi_X(0)\), in the usual case.

If \((X, Y)\) has a binormal distribution standard normal margins and correlation coefficient then \(Z = aX + bY\) has a normal distribution \(N(x/\sigma(a,b))\) with \(\sigma(a,b) = a^2 + b^2 + 2 \rho (a \ b)\).

If \(\rho = 0\) (independence), \(\sigma(a,b) = 1\) iff \(a^2 + b^2 = 1\); in the case \(C(X, Z) = a\).

\[
\sum_{n=1}^{\infty} (\prod_{i=1}^{\infty} \varphi_i(t))
\]

\[
\sum_{n=1}^{\infty} F_i(x)
\]

If \(\{X_i\}\) are i.i.d. there “sometimes” exist \((\lambda_k, \delta_k > 0)\) such that
\[
\text{Prob} \{(M_k - \lambda_n)/\delta_n \leq x\} = F^k(\lambda_k + \delta_k \ x) \rightarrow L(x), \ L(x) \text{ then being } \Psi_\alpha(x), \Lambda(x) \text{ or } \Phi_\alpha(x) ; \text{ for } \Lambda(x) \text{ we have } n(1 - F(\lambda_k)) \rightarrow 1, k(1 - F(\lambda_k + \delta_k)) \rightarrow e^{-1} \text{ or } \delta_n \sim 1/n F'(\lambda_n) ; \text{ there are corresponding results for } \Phi_\alpha \text{ and } \Phi_\alpha^*.

If \((X, Y)\) has a bivariate distribution with reduced Gumbel margins then \(Z = \max(X - a, Y - b)\) has a Gumbel distribution \(\Lambda(z - \lambda(a, b))\) with \(\lambda(a, b) = \log \{(e^{-a} + e^{-b})k(b - a)\}\)

If \(k(w) = 1\) (independence), \(\lambda(a, b) = 0\) if \(e^{-a} + e^{-b} = 1\); in that case \(\text{Prob} \{Z \leq X - a\} = \text{Prob} \{X - b \leq X - a\} = e^{-a}\).